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A probabilistic approach to parallel dynamics for the Little–Hopfield model

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Abstract. We present new results on a probabilistic approach to parallel dynamics of the Little–Hopfield model. We propose a truncated auxiliary dynamics method to control a feedback noise in this symmetrical neural network with full connection. It allows us to propose an ansatz for derivation of the explicit recurrence relations for the main and residual (noisy) overlaps evolution for arbitrary discrete moment t.

1. Introduction

In this paper we derive the explicit equations for the main and residual overlaps evolution generated by the parallel dynamics in the case of the Little–Hopfield (LH) model [1, 2]. The fundamental difficulty of the problem is a strong feedback which is an attribute of fully connected symmetrical neural networks, like the LH model, for discussion see e.g. [3].

It is known that one can exactly and explicitly calculate an evolution for the main overlap if this feedback is suppressed either by modification of the model (e.g. feedforward layered neural networks [4–7]) or by the extreme asymmetric dilution of the LH model [8]. In the recent paper [9] it is shown that the extreme dilution allows one to control a residual feedback exactly even for the symmetrical case.

There are several approaches to take into account the long-term temporal correlations in the LH model treating the feedback influence on the network evolution as an intrinsic noise. The first attempt was to model it by a steady Gaussian noise with zero mean and variance $D = \alpha$ [10] which corresponds to the complete neglecting of correlations. Here $\alpha = M/N$, where M is the number of random independent patterns stored in the network and N is the number of neurons. The next one [11, 12] was to suppose that the variance of this Gaussian noise evolves with time as a function of the main overlap. Further natural generalization is an ansatz with non-Gaussian noise [14]. But application of these approximate treatments to the analysis of the long-time evolution of the LH network is not very reliable. Therefore, it is important to calculate its development in time without approximations. In the present paper we also did not succeed in complete resolving of this problem. We propose a new approach to the analysis of the parallel dynamics in the LH network that reproduces the previous results [13, 15] and elucidates the approximations formulated in [14], cf (45) and [14, equations (4), (5)].

The control of the increasing complexity of a feedback noise for networks with full connection is a rather difficult problem, e.g. for the LH model only a few first steps of the main overlap evolution for the parallel dynamics were known [13, 15, 16]. The *analytical* treatments [15, 17] allow one to calculate the explicit equations for the first two steps of the main overlap evolution. The difficulties there manifest themselves as a fast ill-controlled increasing of the number of auxiliary order parameters in the saddle-point calculations. Note that the expressions proposed in [13] correspond to the second-step formulas of the systematic treatments [15-17].

In [16], we proposed a probabilistic approach to the parallel dynamics in the LH model. We demonstrated that the feedback noise can be exactly taken into account via stochastic equations for the evolution of the residual overlaps. This method allows one to rederive the Gardner–Derrida–Mottishaw second step formula for the main overlap and to go on to the next steps. For example, we calculated the third step formula and discussed a dynamical status of the Amit–Gutfreund–Sompolinsky formula for the main overlap which is obtained by the methods of the equilibrium statistical mechanics [18, 19]. The application of our method to the feedforward layered neural networks [4–7] allows one to derive rigorously the system of recursion relations for the main and residual overlaps [5], which gives an exact solution of this model discovered in [6, 7]. Using the same approach we show [9] that an extremely diluted version of the LH model, proposed in [8], can be solved exactly for the case of the symmetrical synaptic connections. The feedback noise in this case can be entirely controlled.

The aim of the present paper is twofold. At first we improve our method by a new trick: a *truncated auxiliary dynamics*. It makes our line of reasoning shorter and more clear. Thereupon, we apply this approach to derivation of an explicit equation for an arbitrary step of the main and residual overlaps evolution generated by the parallel dynamics for the LH model.

We derive them by induction starting, for simplicity, with the zero-temperature case $\theta = 0$. Therefore, in section 2 we introduce necessary notation and definitions and derive the corresponding equations for the first step $t = 1$. Truncated auxiliary dynamics is introduced in section 3. Then, we use this trick to derive recurrences for $t = 2$. In section 4 we start with derivation of the explicit formulas for the main and residual overlaps for $t = 3$ which we use then for induction. There the essential point is the ansatz about the absence of correlations between the Gaussian and 'memory-like' discrete parts of the noise $\{`\alpha\text{'} - \lim v^q_{[N]\setminus i}(t)\}$ for $t \geq 2$, see (45). This ansatz allows one to close the induction in derivation of the explicit equations for the main overlap evolution. The absence of correlations in (45) for $t = 1$ (see (28)) is a consequence of (23)-(27) and of the choice of the initial configuration (8). This leads to the famous second step formula (29) for the main overlap. At present, we have no convincing arguments in favour of (as well as against) this ansatz for $t \geq 2$. The rest of this section is devoted to calculations for the arbitrary step t. In section 5 we generalize the above results for $\theta \neq 0$ and conclude by some remarks.

2. The problem and the first step for the main overlap

The LH network is considered as a model of content-addressable memory, see [1, 2] and the recent book [19]. It is able:

(i) to store extensively many uncorrelated patterns $\{\xi^p\}_{p \in [M]}$, $[M] \equiv \{1, 2, \ldots, M\}$, encoded by the binary code, i.e., $\xi^p \equiv \{\xi^p_i = \pm 1\}_{i \in [N]}$, $p \in [M]$ are independent identically distributed random variables (IIDRV) with $\Pr\{\xi^p_i = \pm 1\} = \frac{1}{2}$, and we are interested in 'α'-$\lim(\bullet) = \lim_{N \to \infty; M = \alpha N}(\bullet)$;

(ii) to retrieve them from a noisy stimulus (almost perfectly for a small α) as attractors in the configurational space of the network of the two-state neurons $\{s_i = \pm 1\}_{i\in[N]}$ interconnected by the synaptic couplings $\{J_{ij}^M\}_{i,j\in[N]}$.

These fundamental features of the LH model are ensured by the Hebbian symmetrical synaptic connections

$$J_{ij}^M = \frac{1}{N}\sum_{p\in[M]} \xi_i^p \xi_j^p \qquad J_{ii}^M = 0;\ i, j \in [N] \tag{1}$$

and by appropriate dynamics for updating the neurons at discrete time intervals, see e.g. [19]. One simple type of synchronous update of the neutrons $\mathscr{D}_i^{(\theta)}: s_i(t) \to s_i(t+1)$, $i \in [N]$, can be described by the transition (conditional) probability

$$\Pr\{s_i(t+1)|\{s_j(t)\}_{j\in[N]}\} = \frac{\exp[\beta s_i(t+1)h_i(t)]}{2\cosh \beta h_i(t)} \tag{2}$$

where $h_i(t) = \Sigma_{j\in[N]\setminus i} J_{ij}^M s_j(t)$. This is parallel dynamics for the temperature $\theta = \beta^{-1}$ (Glauber dynamics, see e.g. [19]), which we use throughout the present paper. But for simplicity we start below with the zero-temperature $\beta = \infty$ and postpone the case of $\beta < \infty$ until section 5.

Following the papers [5, 16] we consider simultaneous evolution of the main:

$$m^q(t) = \text{`}\alpha\text{'-lim}\ m_{[N]}^q(t)\left(\equiv \frac{1}{N}\sum_{i\in[N]} \xi_i^q s_i(t)\right) \tag{3}$$

and the residual overlaps

$$r^p(t) = \text{`}\alpha\text{'-lim}\ r_{[N]}^p(t)\left(\equiv \frac{1}{\sqrt{N}}\sum_{i\in[N]} \xi_i^p s_i(t)\right) \qquad p \in [M]\setminus q \tag{4}$$

which are induced by the parallel zero-temperature dynamics $\mathscr{D}_i^{(\theta=0)}$ (see (2))

$$s_i(t+1) = \operatorname{sign}\left[\sum_{j\in[N]\setminus i} J_{ij}^M s_j(t)\right]. \tag{5}$$

Then, by definitions (1) and (3)–(5) one gets

$$\begin{aligned} m_{[N]}^q(t+1) &= \frac{1}{N}\sum_{i\in[N]} \operatorname{sign}[m_{[N]\setminus i}^q(t) + \xi_i^q v_{[N]\setminus i}^q(t)] \\ r_{[N]}^p(t+1) &= \frac{1}{\sqrt{N}}\sum_{i\in[N]} \operatorname{sign}\left[\frac{1}{\sqrt{N}} r_{[N]\setminus i}^p(t) + \xi_i^p w_{[N]\setminus i}^{q,p}(t)\right] \end{aligned} \tag{6}$$

where we introduce

$$\begin{aligned} v_{[N]\setminus i}^{\mu_1,\mu_2,\ldots,\mu_k} &\equiv \frac{1}{\sqrt{N}}\sum_{p\in[M]\setminus\mu_1,\mu_2,\ldots,\mu_k} \xi_i^p r_{[N]\setminus i}^p(t) \\ w_{[N]\setminus i}^{q,\mu_1,\ldots,\mu_k}(t) &\equiv \xi_i^q m_{[N]\setminus i}^q(t) + v_{[N]\setminus i}^{q,\mu_1,\ldots,\mu_k}(t) \end{aligned} \tag{7}$$

to distinguish the case of an arbitrary set $\{\mu_i\}_{i=1}^k$ from the case of $q \cup \{\mu_i\}_{i=1}^k$.

Let the initial condition $\{s_i(t=0)\}_{i\in[N]}$ be IIDRV correlated with only one pattern ξ^q, i.e.

$$\Pr\{s_i(0)\xi_i^p = \pm 1\} = \tfrac{1}{2}(1 \pm \delta_{p,q} m^q(0)) \qquad m^q(0) \neq 0. \tag{8}$$

Then one can show [16] that the central limit theorem (CLT) is applicable to the random variables $v^q_{[N]\setminus i}(t=0)$ and one gets:

$$\text{`}\alpha\text{'-lim } v^q_{[N]\setminus i}(t=0) \stackrel{d}{=} \sqrt{\alpha}\, \mathcal{N}_i(0,1) \tag{9}$$

where d means convergence in distribution and $\mathcal{N}(a, b)$ is a Gaussian random variable (with mean a and variance b). Due to the structure of $v^q_{[N]\setminus i}(t=0)$ variables ξ^q_i an $v^q_{[N]\setminus i}$ are independent. As a corollary, by (7)–(9) and the strong law of large numbers (SLLN) for the arithmetical mean in (6), one gets for the main overlap (3) the well known formula [10, 16]:

$$m^q(t=1) = \boldsymbol{E}\,\text{sign}[m^q(t=0)+\sqrt{\alpha}\,\mathcal{N}(0,1)] = \text{erf}\left[\frac{m^q(t=0)}{\sqrt{\alpha}}\right] \tag{10}$$

where $\text{erf}\, z = \sqrt{2/\pi}\int_0^z \mathrm{d}x \exp(-x^2/2)$.

By initial conditions, $\{\xi^p_i s_i(0)\}_{i\in[N]}$ are IIDRV with expectations $\boldsymbol{E}(\xi^p_i s_i(0)) = \delta_{p,q} m^q(0)$, see (8). Then by the CLT the residual overlaps (4), at the moment $t=0$, are independent Gaussian noises:

$$r^p(t=0) = \text{`}\alpha\text{'-lim } r^p_{[N]}(t=0) \stackrel{d}{=} \mathcal{N}(0,1) \qquad p\in[M]\setminus q. \tag{11}$$

Therefore, the one-step evolution of the residual overlaps is described by stochastic recursion relations (6) in the 'α'-lim, see (21) below.

3. Truncated auxiliary dynamics

By the full connection of the LH network, any two neurons have the direct interaction between each other. Consequently, for all $i\in[N]$ expressions

$$\text{sign}\left[\frac{1}{\sqrt{N}} r^p_{[N]\setminus i}(t) + \xi^p_i w^{q,p}_{[N]\setminus i}(t)\right] \equiv \xi^p_i s_i(t+1)$$

contain the term $(1/\sqrt{N}) r^p_{[N]\setminus i}(t)$ in common. This term is almost independent of i in the sense that $|r^p_{[N]\setminus i}(t) - r^p_{[N]\setminus j}(t)| \leqslant 2/\sqrt{N}$ for arbitrary $i, j\in[N]$. Therefore, random variables $\{\xi^p_i s_i(t)\}_{i\in[N]}$ at $t\geqslant 1$ are dependent, and we cannot directly apply the CLT to the residual overlap (4) at $t\geqslant 1$. It has to be emphasized that this dependence is created iff one takes into acount the term $(1/\sqrt{N}) r^p_{[N]\setminus i}(t)$.

Let us define truncated auxiliary dynamics, cf. (5) and (7), by

$$s^p_i(t+1) = \text{sign}[w^{q,p}_{[N]\setminus i}(t)] \tag{12}$$

where the influence of the common residual noise $(1/\sqrt{N}) r^p_{[N]\setminus i}(t)$ corresponding to the pattern $p(\neq q)$ is cancelled. Then (cf. [11])

$$r^p_{[N]}(t=1) = \frac{1}{\sqrt{N}} \sum_{i\in[N]} \xi^p_i s^p_i(t=1) + \frac{1}{\sqrt{N}} \sum_{i\in[N]} \xi^p_i [s_i(t=1) - s^p_i(t=1)]. \tag{13}$$

Now, by the CLT for the first term in (13) one gets

$$\text{`}\alpha\text{'-lim } \frac{1}{\sqrt{N}} \sum_{i\in[N]} \xi^p_i s^p_i(t=1) \stackrel{d}{=} \mathcal{N}(0,1) \qquad p\in[M]\setminus q. \tag{14}$$

By (7)–(9) we get for the density of the probability distribution

$$F^{(0)}_w(x) = \text{`}\alpha\text{'-lim } \Pr\{w^{q,p}_{[N]\setminus i}(t=0) \leqslant x\}$$

the following expression:

$$p_{w,0}(x)=\frac{\mathrm{d}}{\mathrm{d}x}F_w^{(0)}(x)=\frac{1}{2\sqrt{2\pi\alpha}}\sum_{\sigma=\pm1}\exp\left[-\frac{(x-\sigma m^q(0))^2}{2\alpha}\right]. \tag{15}$$

The independence of the random variables ξ_i^p and $w_{[N]\setminus i}^{q,p}(t=0)$ implies that $F_{\xi w}^{(0)}(x)=$ 'α'-$\lim\{\xi_i^p w_{[N]\setminus i}^{q,p}(t=0)\leq x\}=F_w^{(0)}$, i.e. one gets

$$\frac{\mathrm{d}}{\mathrm{d}x}F_{\xi w}^{(0)}=p_{w,0}(x) \tag{16}$$

Due to the independence of the random variables $r_{[N]\setminus i}^p(t=0)$ and $\xi_i^p w_{[N]\setminus i}^{q,p}(t=0)$ we have for conditional probabilities

$$\Pr\{\xi_i^p w_{[N]\setminus i}^{q,p}(t=0)\in A|r_{[N]\setminus i}^p(t=0)\}=\Pr\{\xi_i^p w_{[N]\setminus i}^{q,p}(t=0)\in A\}.$$

Then, comparing

$$\xi_i^p s_i(t=1)=\mathrm{sign}\left[\frac{1}{\sqrt{N}}r_{[N]\setminus i}^p(t=0)+\xi_i^p w_{[N]\setminus i}^{q,p}(t=0)\right]$$

see (2), with $\xi_i^p s_i^p(t=1)=\mathrm{sign}[\xi_i^p w_{[N]\setminus i}^{q,p}(t=0)]$, cf (12), taking into account that

$$\Pr\{r_{[N]\setminus i}^p(t=0)=\mathrm{o}(\log N),\ N\to\infty\}=1$$

and using (16), we obtain for the random variable equal to the conditional probability of $\xi_i^p(s_i(t=1)-s_i^p(t=1))\equiv\zeta_i^p=\{0;\pm2\}$ given σ-algebra $\Delta_{r^p(0)}$ generated by $r_{[N]\setminus i}^p(t=0)$, the following representation $(N\to\infty)$:

$$\begin{aligned}\Pr\{\zeta_i^p=\pm2|r_{[N]\setminus i}^p(t=0)\}&\\ &=\theta(\pm r_{[N]\setminus i}^p(t=0))\Pr\left\{\xi_i^p w_{[N]\setminus i}^{q,p}(t=0)\in\left[0;\frac{1}{\sqrt{N}}|r_{[N]\setminus i}^p(t=0)|\right]\right\}\\ &=\frac{|r_{[N]\setminus i}^p(t=0)|}{\sqrt{N}}p_{w,0}(0)\theta(\pm r_{[N]\setminus i}^p(t=0))+\mathrm{o}\left(\frac{1}{\sqrt{N}}\right).\end{aligned} \tag{17}$$

where $\theta(x)$ is a Heaviside step function.

Note that for the sequence of IIDRV $\{\gamma_i=0,1\}_{i=1}^N$ with $\Pr(\gamma_i=1)=c/\sqrt{N}$ we have the SLLN in the following form $(N\to\infty)$:

$$\frac{1}{\sqrt{N}}\sum_{i=1}^N(\gamma_i-\boldsymbol{E}\gamma_i)\overset{\Pr=1}{=}\mathrm{o}(N^{-(1/4)+\varepsilon}). \tag{18}$$

for arbitrary $\varepsilon>0$. The same property (18) holds also in the case of the weak dependence. For instance, for the random variables ζ_i^p restricted onto $\Delta_{r^p(0)}$, i.e., for random variables $\zeta_i^p I_{A_k}(\cdot)$ with probability distribution $P(\zeta_i^p|A_k)$, where $I_{A_k}(\cdot)$ is indicator of A_k and $\{A_k\}_k$ are atoms of the σ-algebra $\Delta_{r^p(0)}$. Therefore, we get

$$\text{'}\alpha\text{'-}\lim\frac{1}{\sqrt{N}}\sum_{i=1}^N(\zeta_i^p-\boldsymbol{E}(\zeta_i^p|r_{[N]\setminus i}^p(t=0)))=0\quad(P-\text{a.s.}) \tag{19}$$

Then (17) and (19) give for $N\to\infty$:

$$\frac{1}{\sqrt{N}}\sum_{i\in[N]}\xi_i^p(s_i(t=1)-s_i^p(t=1))=2p_{w,0}(0)r_{[N]\setminus i}^p(t=0)+\mathrm{o}(1) \tag{20}$$

and, together with (14), (15), we obtain from (13) that

$$r^p(t=1)=\text{'}\alpha\text{'-}\lim r_{[N]\setminus i}^p(t=1)\overset{d}{=}\mathcal{N}^{(p)}(0,1)+2p_{w,0}(0)r^p(t=0). \tag{21}$$

Remark 3.1. Since the random variables $1/\sqrt{N}\,\Sigma_{j\in[N]}\,\xi_j^p s_j^p(t=1)$ and $r_{[N]}^p(t=0)$ are dependent the same is also valid for the variables $\mathcal{N}^{(p)}(0,1)$ and $r^p(t=0)$ in the stochastic recurrence relation for the limiting residual overlaps (21), see below (25).

As above, to calculate $m^q(t=2)$ we have to obtain distribution for the limiting random variables 'α'-lim $v_{[N]\setminus i}^q(t=1)$, see (6), (9). First, we consider variance $D(t=1)=\mathrm{Var}(r^p(t=1))$. Then using (14) and (21) we can rewrite it in the following form:

$$D(t=1)=\text{'}\alpha\text{'-lim Var}\left[\frac{1}{\sqrt{N}}\sum_{i\in[N]}\xi_i^p(s_i^p(t=1)+2p_{w,0}(0)s_i(t=0))\right]. \tag{22}$$

Now, $s_i^p(t=1)s_i(t=0)=\mathrm{sign}[s_i(t=0)\xi_i^q m_{[N]\setminus i}^q(t=0)+s_i(t=0)v_{[N]\setminus i}^{q,p}(t=0)]$, see (7) and (12). Then independence of variables $s_i(t=0)$ and $v_{[N]\setminus i}^{q,p}(t=0)$, together with (8), (9), implies that

$$\text{'}\alpha\text{'-lim}\ \boldsymbol{E}(s_i^p(t=1)s_i(t=0))=m^q(t=0)m^q(t=1). \tag{23}$$

Similarly, one gets that

$$\begin{aligned}\boldsymbol{E}(s_i^p(t=1)s_j(t=0))&=\boldsymbol{E}(s_i(t=1)s_j(t=0))\\&=\boldsymbol{E}(s_i^p(t=1)s_j^p(t=0))=0\qquad i\neq j.\end{aligned}$$

Then from (22), (23) we obtain:

$$D(t=1)=1+(2p_{w,0}(0))^2+4m^q(t=0)m^q(t=1)p_{w,0}(0) \tag{24}$$

and in particular (see (21) and remark 3.1):

$$\mathrm{cov}(\mathcal{N}^{(p)}(0,1),r^p(t=0))=m^q(t=0)m^q(t=1). \tag{25}$$

Using equation (13), (14) and (20), (21), we can represent the residual overlaps for $t=1$ (see (6), (7)) as follows:

$$r_{[N]\setminus i}^p(t=1)=\frac{1}{\sqrt{N}}\sum_{j\in[N]\setminus i}\xi_j^p s_j^p(t=1)+2p_{w,0}(0)\frac{1}{\sqrt{N}}\sum_{j\in[N]}\xi_j^p s_j(t=0)+\mathrm{o}(1). \tag{26}$$

Note that subtraction of the term $(1/\sqrt{N})\xi_i^p s_i(t=1)$ from the $r_{[N]}^p(t=1)$ (cf. (13), (20) and left-hand side of (26)) we represent in the form with the second sum over $[N]$. Correction corresponding to the term $j=i$ in the last sum in (26) has the same order as o(1) and could be dropped. But we save it in (26) because it is this term that creates dependence of the random variables $\{\xi_i^p r_{[N]\setminus i}^p(t=1)\}_{p(\neq q)}$. This is the origin of non-Gaussian distribution for the variable $v_{[N]\setminus i}^q(t)$, since from (7) and (26) we obtain

$$\begin{aligned}v_{[N]\setminus i}^q(t=1)=&\frac{1}{\sqrt{N}}\sum_{p\in[M]\setminus q}\xi_i^p\frac{1}{\sqrt{N}}\sum_{j\in[N]\setminus i}\xi_j^p[s_j^p(t=1)+2p_{w,0}(0)s_j(t=0)]\\&+\frac{M-1}{N}2p_{w,0}(0)s_i(t=0)+\mathrm{o}(1).\end{aligned} \tag{27}$$

Now we can apply the CLT to the double sum in (27). Then by (22) and (24) one gets

$$\text{'}\alpha\text{'-lim}\ v_{[N]\setminus i}^q(t=1)\stackrel{d}{=}\sqrt{\alpha}\mathcal{N}^{(q)}(0,D(t=1))+2\alpha p_{w,0}(0)s_i(t=0). \tag{28}$$

Hence, by the SLLN for the arithmetical mean in (6) together with (28) we get for the main overlap at $t=2$ another famous formula [13, 15, 16]:

$$m^q(t=2)=\sum_{\sigma_0=\pm1}\frac{1+\sigma_0 m^q(t=0)}{2}\,\mathrm{erf}\left[\frac{m^q(t=1)+2\sigma_0\alpha p_{w,0}(0)}{\sqrt{\alpha D(t=1)}}\right]. \tag{29}$$

Remark 3.2. Formula (29) was discovered in [15], see also [13, 17]. The method which we exploited in [16] allowed one to check (29) and to go further, e.g. to the explicit formula for $m^q(t=3)$. The truncated auxiliary dynamics trick (12) and a new representation for the residual overlaps (cf (6) and (26)) improves the method. It allows one to simplify the derivation of the explicit structure of the formula for the main overlap evolution for the aribtrary moment t, see section 4.

4. Recursions for the main and residual overlaps

Before proceeding to the induction for the general case, it is useful to explain our strategy for the particular example $m^q(t=3)$.

First one has to calculate the residual overlaps $r^p(t=2)$, see (6). Using the truncated auxiliary dynamics (12) and representation (13), we get

$$\begin{aligned} \xi_i^p s_i(t=2) &= \operatorname{sign}\left[\frac{1}{\sqrt{N}}\, r^p_{[N]\setminus i}(t=1) + \xi_i^p w^{q,p}_{[N]\setminus i}(t=1)\right] \\ \xi_i^p s_i^p(t=2) &= \operatorname{sign}[\xi_i^p w^{q,p}_{[N]\setminus i}(t=1)]. \end{aligned} \tag{30}$$

and

$$\text{`}\alpha\text{'-lim}\,\frac{1}{\sqrt{N}}\sum_{i\in[N]} \xi_i^p s_i^p(t=2) \stackrel{d}{=} \mathcal{N}^{(p)}(0,1). \tag{31}$$

Therefore, to calculate the 'α'-lim for the second term in representation (13) for $t=2$ one has to obtain the distribution $F_w^{(1)}$ for the noise 'α'-lim $w^{q,p}_{[N]\setminus i}(t=1)$ in (30). According to (7) and (28) we obtain

$$\begin{aligned} &\text{`}\alpha\text{'-lim}\; w^{q,p}_{[N]\setminus i}(t=1) \\ &\qquad = \xi_i^q m^q(t=1) + 2\alpha p_{w,0}(0) s_i(t=0) + \sqrt{\alpha}\,\mathcal{N}^{(q)}(0, D(t=1)). \end{aligned} \tag{32}$$

To take into account correlations between variables ξ_i^q and $s_i(t=0)$ (see (8)), then the distribution density $(d/dx)F^{(1)}_{\xi w}(x)$ for the random variable 'α'-lim $\xi_i^p w^{q,p}_{[N]\setminus i}(t=1)$ in (30) gets the form (cf (15), (16)):

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} F^{(1)}_{\xi w}(x) = p_{w,1}(x) &= \frac{1}{\sqrt{2\pi\alpha D(t=1)}} \sum_{\sigma_1,\sigma_2=\pm 1} \frac{1+\sigma_1\sigma_2 m^q(t=0)}{4} \\ &\quad\times \exp\left[-\frac{(x-\sigma_1 m^q(t=1) - 2\sigma_2\alpha p_{w,0}(0))^2}{2\alpha D(t=1)}\right]. \end{aligned} \tag{33}$$

Now, using the same line of reasoning as in (17)–(20), by (30) and (33) we obtain (cf (20), 21)):

$$r^p(t=2) \stackrel{d}{=} \mathcal{N}^{(p)}(0,1) + 2p_{w,1}(0) r^p(t=1). \tag{34}$$

Similar to the case of $t=1$, see (26), it is convenient to rewrite (34) (see (14), (21) and (31)) in the following form (cf (22)):

$$\begin{aligned} r^p_{[N]\setminus i}(t=2) &= \frac{1}{\sqrt{N}} \sum_{j\in[N]\setminus i} \xi_j^p[s_j^p(t=2) + 2p_{w,1}(0)(s_j^p(t=1) + 2p_{w,0}(0)s_j(t=0))] \\ &\quad + \frac{2}{\sqrt{N}} p_{w,1}(0)\xi_i^p[s_i^p(t=1) + 2p_{w,0}(0)s_i(t=0)] + o(1). \end{aligned} \tag{35}$$

Here we again have to save the second term in (35), see (26), (27) and the comments after (26). Then by (7), see also (27), we have

$$v^q_{[N]\setminus i}(t=2)=2p_{w,1}(0)\left[\frac{1}{N}\sum_{p\in[M]\setminus q} s^p_i(t=1)+\frac{M-1}{N}2p_{w,0}(0)s_i(t=0)\right]$$
$$+\frac{1}{\sqrt{N}}\sum_{p\in[M]\setminus q}\frac{\xi^p_i}{\sqrt{N}}$$
$$\times\sum_{j\in[N]\setminus i}\xi_j[s^p_j(t^p=2)+2p_{w,1}(0)(s_j(t^p=1)+2p_{w,0}(0)s_j(t=0))]+o(1). \quad (36)$$

As above, see (27), we can apply the CLT to the first term in (36). Then in 'α'-lim it converges to $\sqrt{\alpha}\,\mathcal{N}^{(q)}(0, D(t=2))$, where, by (35) and (36), $D(t=2)=\mathrm{Var}(r^p(t=2))$, cf. (22). In this limit we get also

$$\text{'}\alpha\text{'-lim}\,\frac{1}{M}\sum_{p\in[M]\setminus q}(s^p_i(t=1)-s_i(t=1)=0 \qquad (\mathrm{Pr}=1)$$

see (17). Hence, cf (28), one gets

$$\text{'}\alpha\text{'-lim}\,v^q_{[N]\setminus i}(t=2)$$
$$\overset{d}{=}\sqrt{\alpha}\,\mathcal{N}^{(q)}(0, D(t=2))+2\alpha p_{w,1}(0)[s_i(t=1)+2p_{w,0}(0)s_i(t=0)]. \quad (37)$$

To calculate the variance

$$D(t=2)=1+(2p_{w,1}(0))^2D(t=1)+4p_{w,1}(0)\,\mathrm{cov}(\mathcal{N}^{(p)}(0,1), r^p(t=1)) \quad (38)$$

see (34), we can use the same line of reasoning as in (22)–(25). Then by (35) one gets

$$D(t=2)=\text{'}\alpha\text{'-lim}\,\frac{1}{N}$$
$$\times\sum_{j\in[N]}\mathrm{Var}[s^p_j(t=2)+2p_{w,1}(0)(s^p_j(t=1)+2p_{w,0}(0)s_i(t=0))]$$
$$=1+(2p_{w,1}(0))^2D(t=1)$$
$$+4p_{w,1}(0)\boldsymbol{E}[s^p_j(t=2)(s^p_j(t=1)+2p_{w,0}(0)s_i(t=0))]. \quad (39)$$

Taking into account (28), (29), the representation (32) and

$$\text{'}\alpha\text{'-lim}\,\mathrm{Pr}\{\xi^q_j s^p_j(t=1)=\pm1\}=\frac{1\pm m^q(t=1)}{2}$$

we obtain

$$\text{'}\alpha\text{'-lim}\,\boldsymbol{E}(s^p_j(t=2)s^p_j(t=1))=m^q(t=1)m^q(t=2). \quad (40)$$

By the same reasoning, we get

$$\text{'}\alpha\text{'-lim}\,\boldsymbol{E}(s^p_j(t=2)s^p_j(t=0))$$
$$=m^q(t=0)m^q(t=2)+\frac{1-(m^q(t=0))^2}{2}$$
$$\times\sum_{\sigma=\pm1}\sigma\,\mathrm{erf}\left(\frac{m^q(t=1)+2\sigma\alpha p_{w,0}(0)}{\sqrt{\alpha D(t=1)}}\right). \quad (41)$$

Therefore, according to (38)-(41) we obtain, cf. (25),

$$\operatorname{cov}(\mathcal{N}^{(p)}(0,1), r^p(t=1))$$
$$= m^q(t=1)m^q(t=2)+2p_{w,0}(0)\bigg[m^q(t=0)m^q(t=2)$$
$$+\frac{1-(m^q(t=0))^2}{2}\sum_{\sigma=\pm1}\sigma\,\operatorname{erf}\left(\frac{m^q(t=1)+2\sigma\alpha p_{w,0}(0)}{\sqrt{\alpha D(t=1)}}\right)\bigg]. \tag{42}$$

Hence, by the SSLN for the arithmetical mean in (6) together with (37) one gets for the main overlap at $t=3$ the formula derived in [16], cf (29),

$$m^q(t=3)=\sum_{\sigma_0,\sigma_1\pm1}\frac{[1+\sigma_0 m^q(t=0)]}{2}\frac{[1+\sigma_1 m^q(t=1)]}{2}$$
$$\times\operatorname{erf}\left[\frac{m^q(t=2)+2\alpha p_{w,1}(0)(\sigma_1+\sigma_0 2p_{w,0}(0))}{\sqrt{\alpha D(t=2)}}\right] \tag{43}$$

where $D(t=2)$ is defined by (38), (42) and

$$\frac{[1+\sigma_0 m^q(t=0)]}{2}\frac{[1+\sigma_1 m^q(t=1)]}{2}=\Pr\{\xi_i^q s_i(t=0)=\sigma_0;\, \xi_i^q s_i(t=1)=\sigma_1\}.$$

Using the same line of reasoning as above, we can proceed to the general inductive step. Let $r^p_{[N]\setminus i}(t)$ have the form (see (26) and (35) for $t=1,2$):

$$r^p_{[N]\setminus i}(t)=\frac{1}{\sqrt{N}}\sum_{j\in[N]\setminus i}\xi_j^p s_j^p(t)a_0(t)+\frac{1}{\sqrt{N}}\sum_{j\in[N]\setminus i}\xi_j^p\left[\sum_{\tau=0}^{t-1}a_{t-\tau}(t)s_j^p(\tau)\right]+o(1) \tag{44}$$

where $a_0(t)=1$ and $s_j^p(\tau=0)\equiv s_j(\tau=0)$. Then, similarly to (28) and (37), we have

$$`\alpha\text{'-lim}\; v^q_{[N]\setminus i}(t)\overset{d}{=}\sqrt{\alpha}\,\mathcal{N}^{(q)}(0,D(t))+\alpha\sum_{\tau=0}^{t-1}a_{t-\tau}(t)s_i(\tau). \tag{45}$$

Here we again used that 'α'-lim $1/M\,\Sigma_{p\in[M]\setminus q}(s_i^p(t)-s_i(t))=0$ with $\Pr=1$, see (17).

To close the induction we use here the ansatz mentioned in introduction. Namely, we suppose that random variables $\{`\alpha\text{'-lim}\;\xi_i^q v^q_{[N]\setminus i}(t)\}_i$ are the sums of the independent Gaussian and discrete ('memory-like') parts, as it is represented by (45). Hence, one gets for the corresponding distribution density the following:

$$g_t(x)=\frac{1}{\sqrt{2\pi\alpha D(t)}}\sum_{\{\sigma_\tau=\pm1\}_{\tau=0}^{t-1}}P_1\{\sigma_0;\dots;\sigma_{t-1}\}\exp\left[-\frac{(x-\alpha\,\Sigma_{\tau=0}^{t-1}a_{t-\tau}(t)\sigma_\tau)^2}{2\alpha D(t)}\right] \tag{46}$$

where $P_1(\sigma_0,\sigma_1,\dots,\sigma_{t-1})\equiv\Pr\{\xi_i^q s_i(0)=\sigma_0;\dots;\xi_i^q s_i(t-1)=\sigma_{t-1}\}$. Then, again by the SLLN and (6), (46), we obtain for the main overlap (see (29), (43)) general recurrence relation:

$$m^q(t+1)=\sum_{\{\sigma_\tau=\pm1\}_{\tau=0}^{t-1}}P_1(\sigma_0,\sigma_1,\dots,\sigma_{t-1})\operatorname{erf}\left[\frac{m^q(t)+\alpha\,\Sigma_{\tau=0}^{t-1}a_{t-\tau}(t)\sigma_\tau}{\sqrt{\alpha D(t)}}\right]. \tag{47}$$

On the other hand, by definitions (6) and relation (45), one gets representation

$$r^p(t+1)\overset{d}{=}`\alpha\text{'-lim}\frac{1}{\sqrt{N}}\sum_{i\in[N]}\operatorname{sign}\left[\frac{r^p_{[N]\setminus i}(t)}{\sqrt{N}}+\xi_i^p\xi_i^q m^q_{[N]}(t)\right.$$
$$\left.+\sqrt{\alpha}\,\mathcal{N}^{(p)}(0,D(t))+\alpha\sum_{\tau=0}^{t-1}a_{t-\tau}(t)\xi_i^p s_i(\tau)\right]. \tag{48}$$

Using the independence of the first and second terms in (45), we can calculate the distribution density of the variable 'α'-lim $\xi_i^p w_{[N]\setminus i}^{q,p}(t) \overset{d}{=} \xi_i^p \xi_i^q m^q(t) + \sqrt{\alpha}\mathcal{N}^{(p)}(0, D(t)) + \alpha \sum_{\tau=0}^{t-1} a_{t-\tau}(t)\xi_i^p s_i(\tau)$, cf. (15), (16) and (32), (33),

$$p_{w,t}(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_{\xi w}^{(t)}(x)$$

$$= \frac{1}{\sqrt{2\pi\alpha D(t)}} \sum_{\sigma=\pm 1;\{\sigma_\tau=\pm 1\}_{\tau=0}^{t-1}} P_2(\sigma, \sigma_0, \sigma_1, \ldots, \sigma_{t-1})$$

$$\times \exp\left[-\frac{(x - \sigma m^q(t) - \alpha \sum_{\tau=0}^{t-1} a_{t-\tau}(t)\sigma_\tau)^2}{2\alpha D(t)}\right] \tag{49}$$

where $P_2(\sigma, \sigma_0, \ldots, \sigma_{t-1}) \equiv \Pr\{\xi_i^p \xi_i^q = \sigma; \xi_i^p s_i(0) = \sigma_0; \ldots; \xi_i^p s_i(t-1) = \sigma_{t-1}\}$. Then, using the truncated auxiliary dynamics and the representation (13), we get by (49) that (48) can be rewritten as

$$r^p(t+1) \overset{d}{=} \text{'}\alpha\text{'-lim}\left[\frac{1}{\sqrt{N}} \sum_{j\in[N]} \xi_j^p s_j^p(t+1) + 2p_{w,t}(0) r_{[N]}^p(t)\right] \tag{50}$$

cf. (21), (34), or in the following form, see (44),

$$r^p(t+1) \overset{d}{=} \text{'}\alpha\text{'-lim}\left[\frac{1}{\sqrt{N}} \sum_{j\in[N]} \xi_j^p \sum_{\tau=0}^{t-1} a_{t+1-\tau}(t+1) s_j^p(\tau)\right] \tag{51}$$

where

$$a_0(t+1) = 1 \qquad a_{t+1-\tau}(t+1) = 2p_{w,t}(0) a_{t-\tau}(t). \tag{52}$$

Comparing (44) and (51), we see that the latter formula completes the inductive step for the residual overlaps evolution, i.e., one gets (cf. (21), (34))

$$r^p(t+1) \overset{d}{=} \mathcal{N}^{(p)}(0, 1) + 2p_{w,t}(0) r^p(t). \tag{53}$$

Finally, we have to obtain explicit formulas for the variance $D(t) = \mathrm{Var}(r^p(t))$ and the probabilities $P_1(\sigma_0; \ldots; \sigma_t)$, $P_2(\sigma; \sigma_0; \ldots; \sigma_t)$, see (47) and (49).

Using (6) and (45), we get

$$\xi_i^q s_i(t+1) = \mathrm{sign}\left[m^q(t) + \xi_i^q \sqrt{\alpha}\, \mathcal{N}^{(q)}(0; D(t)) + \alpha \sum_{\tau=0}^{t-1} a_{t-\tau}(t) \xi_i^q s_i(\tau)\right].$$

Then we have

$$P_1(\sigma_0; \sigma_1; \ldots; \sigma_{t+1})$$

$$= \Pr\left\{\mathrm{sign}\left[m^q(t) + \xi_i^q \sqrt{\alpha}\, \mathcal{N}^{(q)}(0, D(t)) + \alpha \sum_{\tau=0}^{t-1} a_{t-\tau}(t) \xi_i^q s_i(\tau)\right] = \sigma_{t+1}; \{\xi_i^q s_i(\tau) = \sigma_\tau\}_{\tau=0}^{t}\right\}$$

$$= \Pr\left\{\mathrm{sign}\left[m^q(t) + \xi_i^q \sqrt{\alpha}\, \mathcal{N}^{(q)}(0, D(t)) + \alpha \sum_{\tau=0}^{t-1} a_{t-\tau}(t) \sigma_\tau\right] = \sigma_{t+1}\right\} P_1(\sigma_0; \sigma_1; \ldots; \sigma_t)$$

$$= \frac{1}{\sqrt{2\pi\alpha D(t)}} \int_{-\infty}^{\sigma_{t+1}[m^q(t) + \alpha \sum_{\tau=0}^{t-1} a_{t-\tau}(t)]} \mathrm{d}x \exp\left[-\frac{x^2}{2\alpha D(t)}\right] P_1(\sigma_0; \sigma_1; \ldots; \sigma_t)$$

which coincides with the following recurrence relation:

$$P_1(\sigma_0; \ldots; \sigma_{t+1}) = \frac{1}{2}\left\{1 + \sigma_{t+1}\,\mathrm{erf}\left[\frac{m^q(t) + \alpha \sum_{\tau=0}^{t-1} a_{t-\tau}(t)\sigma_\tau}{\sqrt{\alpha D(t)}}\right]\right\} P_1(\sigma_0; \ldots; \sigma_t) \tag{54}$$

with initial condition

$$P_1(\sigma_0) = \frac{1 + \sigma_0 m^q(t=0)}{2}. \tag{55}$$

The same calculations for $P_2(\sigma; \sigma_0; \sigma_1; \ldots; \sigma_{t+1})$ give recurrence relation

$$P_2(\sigma; \ldots; \sigma_{t+1}) = \frac{1}{2}\left\{1 + \sigma_{t+1}\,\mathrm{erf}\left[\frac{\sigma m^q(t) + \alpha \sum_{\tau=0}^{t-1} a_{t-\tau}(t)\sigma_\tau}{\sqrt{\alpha D(t)}}\right]\right\} P_2(\sigma; \ldots; \sigma_t) \tag{56}$$

wiith initial conditions

$$P_2(\sigma; \sigma_0) = \frac{1 + \sigma\sigma_0 m^q(t=0)}{4} \tag{57}$$

and $P_2(\sigma) = \frac{1}{2}$, see (15).

Now we return to evolution of the variance $D(t)$. From (52) one gets, cf. (24), (38),

$$D(t+1) = 1 + (2p_{w,t}(0))^2 D(t) + 4p_{w,t}(0)\,\mathrm{cov}(\mathcal{N}^{(p)}(0,1), r^p(t)). \tag{58}$$

Using representations (50)–(52), we obtain

$$\mathrm{cov}(\mathcal{N}^{(p)}(0,1), r^p(t)) = \text{`}\alpha\text{'-lim}\,\frac{1}{N}\sum_{i,j\in[N]} E\left[\xi_i^p s_j^p(t+1)\xi_i^p \sum_{\tau=0}^{t} a_{t-\tau}(t) s_i^p(\tau)\right]. \tag{59}$$

If one takes into account that the joint distribution of $s_i^p(t+1)$ and $s_i^p(\tau)$ coincides (in the 'α'-lim) with the one for $s_i(t+1)$ and $s_i(\tau)$, see section 3, then

$$E(s_i^p(t+1)s_i^p(\tau)) = \sum_{\{\sigma_\tau = \pm 1\}_{\tau=0}^{t-1}} \sigma_{t+1}\sigma_\tau P_1(\sigma_0; \sigma_1; \ldots \sigma_{t+1}) \equiv c_{t+1,\tau}. \tag{60}$$

As a consequence, finally get for (58):

$$D(t+1) = 1 + 2(p_{w,t}(0))^2 D(t) + 4p_{w,t}(0) \sum_{\tau=0}^{t} a_{t-\tau}(t) c_{t+1,\tau}. \tag{61}$$

Equations (47), (49), (52)–(57) and (61) give recursions for the main and residual overlaps evolution in the case of the zero-temperature parallel dynamics for the LH model.

5. Conclusion

We start this section with generalization of the recursion relations (47), (49), (52)–(57) and (61) to non-zero temperature parallel dynamics $\mathcal{D}_t^{(\theta)}$ (2). Recall that (2) is equivalent to the replacement of (5) by the stochastic equation

$$s_i(t+1) = \mathrm{sign}\left[\sum_{j\in[N]\setminus i} J_{ij}^M s_j(t) + \eta_i(t)\right]. \tag{62}$$

Here $\{\eta_i(t)\}_{i\in[N],t\geq 0}$ are *unquenched* IIDRV with distribution $\Phi_\beta(x)=\Pr\{\eta_i(t)\leq x\}$, which is equal to

$$\Phi_\beta(x)=\tfrac{1}{2}(1+\tanh\beta x). \tag{63}$$

They represent a heat-bath noise for the temperature $\theta=\beta^{-1}$. As a result we get in the arguments of sign in (6) the additional independent noisy terms $\xi_i^q\eta_i(t)$ and $\xi_i^p\eta_i(t)$, $p\in[M]\backslash q$, respectively, which have the same distribution (63) as $\eta_i(t)$. Therefore, taking into account (62) we obtain for the main overlap $m^q(t)$ (cf (6) and (45)):

$$m^q(t+1)=\text{`}\alpha\text{'-lim}\,\frac{1}{N}\sum_{i\in[N]}\operatorname{sign}\Bigg[m^q_{[N]\backslash i}(t)+\xi_i^{(q)}(\sqrt{\alpha}\,\mathcal{N}^{(q)}(0,D_\theta(t))+\eta_i(t))$$
$$+\alpha\sum_{\tau=0}^{t-1}a^{(\theta)}_{t-\tau}(t)\xi_i^q s_i(\tau)\Bigg].$$

Hence, the distribution of the intrinsic noise is the convolution of the probability distributions of two noises: $\sqrt{\alpha}\,\mathcal{N}^{(q)}(0,D_\theta(t))+\alpha\sum_{\tau=0}^{t-1}a^{(\theta)}_{t-\tau}(t)\xi_i^q s_i(\tau)$ and $\eta_i(t)$. Then the distribution density of the variable 'α'-lim $\xi_i^q[v^q_{[N]\backslash i}(t)+\eta_i(t)]$, cf. (46), gets the form:

$$g_t^{(\theta)}(x)=\frac{1}{\sqrt{2\pi\alpha D_\theta(t)}}\sum_{\{\sigma_\tau=\pm1\}_{\tau=0}^{t-1}}\Pr\{\xi_i^q s_i(0)=\sigma_0;\ldots;\xi_i^q s_i(t-1)=\sigma_{t-1}\}$$
$$\times\int_{-\infty}^{+\infty}\mathrm{d}y\,\exp\left[-\frac{(x-y-\alpha\sum_{\tau=0}^{t-1}a_{t-\tau}(t)\sigma_\tau)^2}{2\alpha D_\theta(t)}\right]\frac{\beta}{2\cosh^2\beta y}. \tag{64}$$

Using the (64) and the above prescriptions (6), we obtain

$$m^q(t+1)=\frac{1}{\sqrt{2\pi\alpha D_\theta(t)}}\sum_{\{\sigma_\tau=\pm1\}_{\tau=0}^{t-1}}P_1^{(\theta)}(\sigma_0;\sigma_1;\ldots;\sigma_{t-1})\int_{R_1}\mathrm{d}x\,\exp\left[-\frac{x^2}{2\alpha D_\theta(t)}\right]$$
$$\times\tanh\left[\beta\left(m^q(t)+\alpha\sum_{\tau=0}^{t-1}\sigma_\tau a^{(\theta)}_{t-\tau}(t)+x\right)\right]. \tag{65}$$

where the weight functions $P_1^{(\theta)}(\sigma_0;\sigma_1;\ldots;\sigma_{t-1})$ have the form (cf. (54))

$$P_1^{(\theta)}(\sigma_0;\ldots;\sigma_{t+1})$$
$$=\frac{1}{2}\left\{1+\frac{\sigma_{t+1}}{\sqrt{2\pi\alpha D_\theta(t)}}\int_{R^1}\mathrm{d}x\,\exp\left[-\frac{x^2}{2\alpha D_\theta(t)}\right]\right.$$
$$\left.\times\tanh\left[\beta\left(m^q(t)+\alpha\sum_{\tau=0}^{t-1}\sigma_\tau a^{(\theta)}_{t-\tau}(t)+x\right)\right]\right\}P_1^{(\theta)}(\sigma_0;\sigma_1;\ldots;\sigma_t) \tag{66}$$

with the initial condition (55). In a similar way, instead of recurrence relation (56) we get

$$P_2^{(\theta)}(\sigma;\sigma_0;\ldots;\sigma_{t+1})$$
$$=\frac{1}{2}\left\{1+\frac{\sigma_{t+1}}{\sqrt{2\pi\alpha D_\theta(t)}}\int_{R^1}\mathrm{d}x\,\exp\left[-\frac{x^2}{2\alpha D_\theta(t)}\right]\right.$$
$$\left.\times\tanh\left[\beta\left(\sigma m^q(t)+\alpha\sum_{\tau=0}^{t-1}\sigma_\tau a^{(\theta)}_{t-\tau}(t)+x\right)\right]\right\}P_2^{(\theta)}(\sigma;\sigma_0;\ldots;\sigma_t) \tag{67}$$

with the initial condition (55). Then the distribution density (49) for non-zero temperature takes the form

$$p_{w,t}^{(\theta)}(x)=\frac{1}{\sqrt{2\pi\alpha D_\theta(t)}}\sum_{\sigma=\pm1;\{\sigma_\tau=\pm1\}_{\tau=0}^{t-1}} P_2^{(\theta)}(\sigma;\sigma_0;\ldots;\sigma_{t-1}) \times\int_{-\infty}^{+\infty}\mathrm{d}y\exp\left[-\frac{(x-y-\sigma m_\theta^q(t)-\alpha\sum_{\tau=0}^{t-1}a_{t-\tau}(t)\sigma_\tau)^2}{2\alpha D_\theta(t)}\right]\frac{\beta}{2\cosh^2\beta y} \quad (68)$$

where instead of (52) we have

$$a_0^{(\theta)}(t+1)=1,\ a_{t+1-\tau}^{(\theta)}(t+1)=2p_{w,t}^{(\theta)}(0)a_{t-\tau}^{(\theta)}(t). \quad (69)$$

Therefore, stochastic recurrence relations for the residual overlaps get the form (cf. (53) and (68))

$$r_\theta^p(t+1)\overset{d}{=}\mathcal{N}^{(p)}(0,1)+2p_{w,t}^{(\theta)}(0)r_\theta^p(t). \quad (70)$$

To close the system of equations for the main and residual overlaps evolution for $\theta\neq0$ we have to complete it by the recurrence for the variance of $r_\theta^p(t+1)$, cf. (60), (61):

$$D_\theta(t+1)=1+(2p_{w,t}^{(\theta)}(0))^2D_\theta(t)+4p_{w,t}^{(\theta)}(0)\sum_{\tau=0}^{t}a_{t-\tau}^{(\theta)}(t)c_{t+1,\tau}^{(\theta)} \quad (71)$$

where

$$c_{t+1,\tau}^{(\theta)}=\sum_{\{\sigma_\tau=\pm1\}_{\tau=0}^{t+1}}\sigma_{t+1}\sigma_\tau P_1^{(\theta)}(\sigma_0;\sigma_1;\ldots;\sigma_{t+1}). \quad (72)$$

Remark that nowhere above we have used an *a priori* averaging over the thermal noise associated with dynamics (62), (63), see e.g. [4]. Our formulas (65)–(72) are a consequence of the same line of reasoning as above for $\theta=0$: we are exploiting the CLT and SLLN in the 'α'-lim. As it is clear from the nature of the noise η (62), they have to coincide with the ones we would get when we first average over the thermal noise η and after goes to the 'α'-lim via the CLT and SLLN for realizations of key patterns. For instance, (65) is equal to the equation

$$m^q(t+1)=\text{'}\alpha\text{'-lim}\frac{1}{N}\sum_{i\in[N]}\xi_i^qE_\eta(s_i(t+1))$$

that is often used as a definition of the main overlap for non-zero temperature.

In paper [16] we proposed arguments in favour of dynamical origination of the Amit-Gutfreund-Sompolinsky formula for the main overlap as a limit for $t\to\infty$. The recurrence relations (47) or (65) do not improve these arguments very much because it is difficult to go to $t\to\infty$ directly in these recurrences. This is a hint that the shape of a possible attractor for the main overlap evolution has a rather complicated structure. On the level of two-step (Gardner–Derrida–Mottishaw) formula (29) this problem was discussed in [13].

Explicit relations (54), (66) allow one to consider another important problem: dynamics of the flipping of individual spins along a time trajectory to attractor, In particular, they describe a transition between two regimes of the temporal sequence $\{\xi_i^q s_i(\tau)\}_{\tau\geq1}$ flips: for $\alpha\to0$ (independent flipping) and for a large α (stabilization of the sequence). We hope to return to these problems elsewhere.

Summarizing, we would like to stress that our approach to derivation of the recurrence relations for evolution of the main and residual overlaps for the LH model is far from being rigorous. But we think that an advantage of the probabilistic approach consists in the possibility to refine upon a systematic analysis of the feedback noise via stochastic equation for the evolution of the residual overlaps. This approach allows one to distinguish an exactly solvable cases (like feedforward and extremely diluted neural networks) and to get hints for approximations like the ansatz for the general inductive step for the main overlap evolution forrmulated in section 4. In this sense expression (45) elucidates the ansatz proposed in [14], see equations (4) and (5). We calculate explicitly the effective Gaussian noise and the discrete ('memory-like') noise involved in (45) but we assume them being independent

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